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THE COMPARABILITY GRAPH OF A MODULAR LATTICE

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The algebraic technique on fundamental chains [6] of Cohen–Macaulay partially ordered sets enables us to show that the comparability graph of a finite modular lattice L of rank $d-1$ is d-connected if there exists a closed interval $[x,y]$ of rank 3 of L with $\mu_L(x,y)\neq 0$, where μ_L is the Möbius function of L.

Introduction

Let P be a finite partially ordered set and $\mathrm{Com}(P)$ its comparability graph [1, p. 328]. If P is Cohen–Macaulay [3] of rank $d-1$, then $\mathrm{Com}(P)$ is $(d-1)$-connected, i.e., the subgraph $\mathrm{Com}(P-W)$ is connected for every subset W of P with $\sharp(W)<d-1$. Here, $\sharp(W)$ is the cardinality of a finite set W. For which Cohen–Macaulay partially ordered sets P of rank $d-1$ is the comparability graph $\mathrm{Com}(P)$ d-connected?

A finite distributive lattice L is called planar if no closed interval of L of rank 3 is a boolean algebra. The comparability graph $\mathrm{Com}(L)$ of a finite non-planar distributive lattice L of rank $d-1$ is d-connected ([11, Theorem (3.3)]). The purpose of the present paper is to extend the idea of [11] to comparability graphs of finite modular lattices and to prove the following

(0.1) Theorem. *Let L be a finite modular lattice of rank $d-1$ (≥ 3) and suppose that there exists a closed interval $I=[x,y]$ of L with $\mathrm{rank}(I)=3$ and with $\mu_L(x,y)\neq 0$, where μ_L is the Möbius function of L. Then, the comparability graph $\mathrm{Com}(L)$ of L is d-connected, that is to say, the subgraph $\mathrm{Com}(L-W)$ is connected for every subset W of L with $\sharp(W)<d$.*

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1. Fundamental chains of modular lattices

We here summarize basic information about algebra and combinatorics on finite modular lattices. We refer the reader to [2] and [10] for elementary definitions and terminology on lattices and partially ordered sets. Every partially ordered set ("poset" for short) to be studied is finite.

A *chain* is a totally ordered set. The *length* of a chain C is $\ell(C) := \sharp(C) - 1$. A totally ordered subset in a poset P is called a chain of P. Let $\mathrm{rank}(P)$ denote the *rank* of a poset P and $\mathrm{rank}_P(x)$ the rank of an element x of P, that is to say, $\mathrm{rank}(P)$ is the maximal length of chains of P and $\mathrm{rank}_P(x)$ is the maximal length of chains of P descending from x. When $x, y \in P$, we say that y *covers* x if $x < y$ and $x < z < y$ for no element $z \in P$. If $x < y$ in a poset P, then the induced subposet $[x, y] := \{z \in P;\ x \leq z \leq y\}$ is called a *closed interval* of P.

Let $\hat{0}$ (resp. $\hat{1}$) denote the unique minimal (resp. maximal) element of a (finite) lattice L. Every closed interval of a lattice is again a lattice. An *atom* of a lattice L is an element which covers $\hat{0} \in L$. We say that a lattice L is *atomic* if every element is the join of atoms of L. A lattice L is called *semimodular* if the following condition is satisfied: If $x, y \in L$ both cover $x \wedge y$, then $x \vee y$ covers both x and y. Every semimodular lattice L is pure, i.e., all maximal chains of L have the same length. A *geometric lattice* is a lattice which is both atomic and semimodular. Moreover, we say that a lattice L is *modular* if, for all elements x, y and z in L with $x \leq z$, we have $x \vee (y \wedge z) = (x \vee y) \wedge z$. Every modular lattice is semimodular. If L is a modular lattice and if $x, y \in L$, then

$$\mathrm{rank}_L(x) + \mathrm{rank}_L(y) = \mathrm{rank}_L(x \wedge y) + \mathrm{rank}_L(x \vee y),$$

and the closed intervals $[x \wedge y, y]$ and $[x, x \vee y]$ are isomorphic ([10, Eq. (6), p. 104] and [2, Theorem 13, p. 13]).

Let μ_P denote the Möbius function [10, p. 116] of a poset P.

(1.1) Lemma. *Let L be a modular lattice. Then, the following conditions are equivalent:*

(i) $\mu_L(\hat{0}, \hat{1}) \neq 0$;

(ii) $\hat{1}$ *is the join of atoms of* L;

(iii) L *is geometric.*

Proof. Suppose that $\mu_L(\hat{0}, \hat{1}) \neq 0$. Then, $\hat{1}$ must be the join of atoms of L ([10, Corollary 3.9.5]). Hence, thanks to [2, Theorem 6, p. 88], [2, Theorem 14, p. 16] together with [2, Theorem 7, p. 89], L is atomic. Moreover, [2, Rota's Theorem, p. 102] guarantees that $\mu_L(\hat{0}, \hat{1}) \neq 0$ if L is a geometric lattice. ∎

Given a finite partially ordered set $P = \{x_1, x_2, \ldots, x_v\}$ of rank $d - 1$, let $A = k[x_1, x_2, \ldots, x_v]$ denote the polynomial ring over a field k, whose indeterminates are the elements of P, with the standard grading, i.e., each $\deg x_i = 1$, and let I_P denote the ideal of A which is generated by those quadratic monomials $x_i x_j$ such that x_i and x_j are incomparable in the partial order of P. Define the linear form $\theta_j \in A$ by

$$\theta_j = \sum_{x\in P,\ \mathrm{rank}_P(x)=j-1} x$$

for each $1 \leq j \leq d$, and let Q_P denote the quotient algebra

$$Q_P = A/(I_P, \theta_1, \theta_2, \cdots, \theta_d),$$

which inherits the graded structure from A of the form $Q_P = \bigoplus_{n=0}^{\infty}[Q_P]_n$. Here, $[Q_P]_n$ is the subspace of Q_P which is spanned over k by all monomials of degree n. Since $x_i^2 = 0$ in Q_P for every $1 \leq i \leq v$, it follows that $[Q_P]_n = 0$ if $n > d$, i.e.,

$$Q_P = [Q_P]_0 \bigoplus [Q_P]_1 \bigoplus \cdots \bigoplus [Q_P]_d.$$

We refer the reader to, e.g., [3, p. 602] for the definition of Cohen–Macaulay posets. See also [4], [5], [8] and [9]. Every semimodular lattice is Cohen–Macaulay. When P is a Cohen–Macaulay poset, we define the subspace $[\mathrm{Soc}(Q_P)]_n$ of $[Q_P]_n$ by

$$[\mathrm{Soc}(Q_P)]_n = \{y \in [Q_P]_n;\ x_i y = 0 \text{ for every } 1 \leq i \leq v\}.$$

We say that $\mathrm{Soc}(Q_P) := \bigoplus_{n=0}^{d}[\mathrm{Soc}(Q_P)]_n$ is the *socle* of Q_P. We write $\dim_k[\mathrm{Soc}(Q_P)]_n$ for the dimension of the vector space $[\mathrm{Soc}(Q_P)]_n$ over k.

Let $\tilde{H}_i(P;k)$ denote the i-th reduced order homology group, e.g., [3, p. 596] of P with coefficient in k.

(1.2) Proposition. ([8, Theorem (5.1)]). *Let P be a Cohen–Macaulay partially ordered set of rank $d-1$. Then,*

$$\dim_k[\mathrm{Soc}(Q_P)]_n = \sum_{W\subset P,\ \sharp(W)=d-n} \dim_k \tilde{H}_{n-1}(P-W;k)$$

for every $1 \leq n \leq d$.

Recall from [6] and [7] that a *fundamental chain* (*of length* s) of a modular lattice L is a chain of L of the form

$$C : \hat{0} = x_0 < x_1 < \cdots < x_{s-1} < x_s = \hat{1}$$

such that

(i) $\mu_L(x_{i-1}, x_i) \neq 0$ for each $1 \leq i \leq s$;

(ii) $\mu_L(x_{i-1}, x_{i+1}) = 0$ for each $1 \leq i < s$.

Define $\mu_L(C)$ by

$$\mu_L(C) = \prod_{i=1}^{s} |\mu_L(x_{i-1}, x_i)|$$

for the above fundamental chain C of L. Let $\mathcal{F}(L;s)$ denote the set of all fundamental chains of length s of L.

(1.3) Proposition. ([6, Theorem (4.8)]). *If L is a modular lattice of rank $d-1$, then*

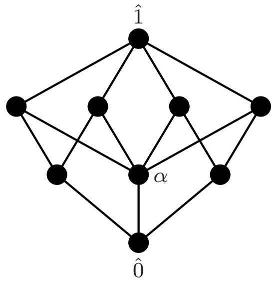
$$\dim_k[\mathrm{Soc}(Q_P)]_n = \sum_{C\in\mathcal{F}(L;d-n-1)} \mu_L(C)$$

for every $1\leq n\leq d$.

(1.4) Corollary. *Let L be a modular lattice of rank $d-1$. Then, the comparability graph $\mathrm{Com}(L)$ of L is d-connected if and only if there exists no fundamental chain of length $d-2$ of L.*

Proof. By virtue of Proposition (1.2), $[\mathrm{Soc}(Q_L)]_1=(0)$ if and only if $\tilde{H}_0(L–W;k)=0$ for every subset W of P with $\sharp(W)=d-1$. Since $\dim_k \tilde{H}_0(L-W;k)$ is one less than the number of connected components of $\mathrm{Com}(L–W)$, we have $\tilde{H}_0(L–W;k)=0$ if and only if the subgraph $\mathrm{Com}(L–W)$ of $\mathrm{Com}(L)$ is connected. Thus, $[\mathrm{Soc}(Q_L)]_1=(0)$ if and only if $\mathrm{Com}(L)$ is d-connected. Moreover, by Proposition (1.3), $[\mathrm{Soc}(Q_P)]_1=(0)$ if and only if $\mathcal{F}(L;d-2)=\emptyset$. Hence, $\mathrm{Com}(L)$ is d-connected if and only if $\mathcal{F}(L;d-2)=\emptyset$ as required. ∎

The concept of fundamental chains may be defined for an arbitrary Cohen–Macaulay poset with $\hat{0}$ and $\hat{1}$. However, the above Corollary (1.4) is false in general. Let P be the Cohen–Macaulay poset of rank 3 drawn below. Then, the chain $\hat{0}<\hat{1}$ of length 1 is the only fundamental chain of P. However, the comparability graph $\mathrm{Com}(P)$ of P is not 4-connected, since the subgraph $\mathrm{Com}(P-\{\hat{0},\hat{1},\alpha\})$ is disconnected.



2. Proof of Theorem (0.1)

Corollary (1.4) together with some combinatorial lemmata below enables us to give a proof of Theorem (0.1).

(2.1) Lemma. *Let L be a modular lattice of rank $d-1$ (≥ 3) and suppose that L possesses a fundamental chain of length $d-2$, say $\hat{0}=x_0<x_1<\cdots<x_{j-1}<x_{j+1}<\cdots<x_{d-2}<x_{d-1}=\hat{1}$ with each $\mathrm{rank}_L(x_i)=i$.*

(a) *Every atom* α *of* L *satisfies* $\alpha \leq x_{j+1}$.

(b) *If* β *is the join of all the atoms of* L, *then* $\text{rank}_L(\beta) \leq 2$.

Proof. (a) Suppose that there exists an atom α of L with $\alpha \not\leq x_{j+1}$ and let q denote the least integer $j+1 < q \leq d-1$ with $\alpha \leq x_q$. If $q \geq j+3$, then $x_{q-2} < x_{q-2} \vee \alpha < x_q$ and $x_{q-2} \vee \alpha \neq x_{q-1}$. Hence, $\mu_L(x_{q-2}, x_q) \neq 0$, a contradiction. Let us assume $q = j+2$. Since $x_{j-1} \vee \alpha$ is an atom of the closed interval $[x_{j-1}, x_{j+2}]$, and since x_{j+1} is the join of atoms in $[x_{j-1}, x_{j+2}]$ because $\mu_L(x_{j-1}, x_{j+1}) \neq 0$, it follows that x_{j+2} $(= x_{j+1} \vee \alpha)$ is the join of atoms of $[x_{j-1}, x_{j+2}]$. Thus, by Lemma (1.1), $\mu_L(x_{j-1}, x_{j+2}) \neq 0$, a contradiction.

(b) Let $\alpha_1 (= x_1), \alpha_2, \ldots, \alpha_n$ denote the atoms of L with $n \geq 3$. Since $\alpha_i \leq x_{j+1}$ for each $1 \leq i \leq n$, we may assume $j \geq 2$. First, we show that there exists $y \in L$ with $x_{j-1} < y < x_{j+1}$ such that $y \neq x_{j-1} \vee \alpha_i$ for each $1 < i \leq n$. In fact, if every element $y \in L$ with $x_{j-1} < y < x_{j+1}$ is of the form $y = x_{j-1} \vee \alpha_i$ for some $1 < i \leq n$, then x_{j+1} is the join of atoms $x_{j-1}, x_{j-2} \vee \alpha_2, \ldots, x_{j-2} \vee \alpha_n$ in the closed interval $[x_{j-2}, x_{j+1}]$. Hence, again by Lemma (1.1), $\mu_L(x_{j-2}, x_{j+1}) \neq 0$, a contradiction.

We now choose $y \in L$ with $x_{j-1} < y < x_{j+1}$ such that $y \neq x_{j-1} \vee \alpha_i$ for each $1 < i \leq n$. Then, $\alpha_i \not\leq y$ for each $1 < i \leq n$. Since $[\hat{0}, \beta]$ is atomic, $y \wedge \beta = \alpha_{i_1} \vee \alpha_{i_2} \vee \cdots \vee \alpha_{i_k}$ for some $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. However, since $\alpha_i \not\leq y$ for each $i > 1$, we have $y \wedge \beta = \alpha_1$. Hence, $\text{rank}_L(x_{j+1}) = \text{rank}_L(y \vee \beta) = \text{rank}_L(y) + \text{rank}_L(\beta) - 1$. Thus, $\text{rank}_L(\beta) = 2$ as desired. ∎

(2.2) Lemma. *If a modular lattice* L *of rank* 4 *possesses a fundamental chain of length* 3, *then no closed interval* I *of* L *with* $\text{rank}(I) = 3$ *is atomic.*

Proof. When I is of the form $I = [\hat{0}, \delta]$ with $\text{rank}_L(\delta) = 3$, the above Lemma (2.1;b) guarantees that I is not atomic. If we apply Lemma (2.1;b) to the dual lattice [10, p. 101] of L, then it follows that $\mu_L(\alpha, \hat{1}) = 0$ for every atom α of L. Hence, the closed interval $I = [\alpha, \hat{1}]$ with an atom α of L is not atomic. Thus, no closed interval I of L with $\text{rank}(I) = 3$ is atomic as required. ∎

(2.3) Lemma. *Let* L *be a modular lattice of rank* $d-1$ (≥ 3) *and suppose that* L *possesses a fundamental chain of length* $d-2$. *Let* α *be an arbitrary atom of* L. *Then, the closed interval* $[\alpha, \hat{1}]$ *possesses a fundamental chain of length* $\geq d-3$.

Proof. Suppose that

$$\hat{0} = x_0 < x_1 < \ldots < x_{j-1} < x_{j+1} < \ldots < x_{d-2} < x_{d-1} = \hat{1}$$

with each $\text{rank}_L(x_i) = i$ is a fundamental chain of length $d-2$ of L. We first assume $2 \leq j \leq d-3$. Let $\alpha \neq x_1$ and write C for the chain

$$\alpha < x_1 \vee \alpha < \cdots < x_{j-1} \vee \alpha < x_{j+2} < x_{j+3} < \cdots < x_{d-2} < x_{d-1} = \hat{1}$$

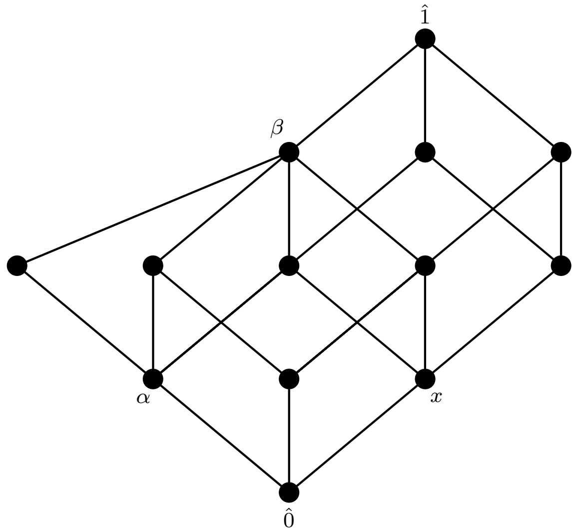
of $[\alpha,\hat{1}]$. The closed interval $[\alpha,x_{j-1}\vee\alpha]$ is a chain of L since $[\alpha,x_{j-1}\vee\alpha]$ is isomorphic to $[\alpha\wedge x_{j-1},x_{j-1}]$ $(=[\hat{0},x_{j-1}])$. Since both the closed intervals $[x_{j-2},x_{j+2}]$ and $[x_{j-1},x_{j+3}]$ are of rank 4 and possess fundamental chains of length 3, it follows from Lemma (2.2) that neither $[x_{j-2}\vee\alpha,x_{j+2}]$ nor $[x_{j-1}\vee\alpha,x_{j+3}]$ is atomic. Hence, $\mu_L(x_{j-2}\vee\alpha,x_{j+2})=0$ and $\mu_L(x_{j-1}\vee\alpha,x_{j+3})=0$. Thus, if $\mu_L(x_{j-1}\vee\alpha,x_{j+2})\neq 0$, then C is a fundamental chain of $[\alpha,\hat{1}]$ of length $d-3$. If $\mu_L(x_{j-2}\vee\alpha,x_{j+1})\neq 0$, then $(C-\{x_{j-1}\vee\alpha\})\cup\{x_{j+1}\}$ is a fundamental chain of $[\alpha,\hat{1}]$ since $\mu_L(x_{j-3}\vee\alpha,x_{j+1})=0$ by Lemma (2.2). If $\mu_L(x_{j-1}\vee\alpha,x_{j+2})=0$ and $\mu_L(x_{j-2}\vee\alpha,x_{j+1})=0$, then $C\cup\{x_{j+1}\}$ is a fundamental chain of $[\alpha,\hat{1}]$.

If $j=1$, then x_2 is the join of all the atoms of L by Lemma (2.1). Thus, since $\mu_L(\alpha,x_4)=0$ by Lemma (2.2), either $\alpha<x_2<x_3<\cdots<x_{d-1}=\hat{1}$ or $\alpha<x_3<x_4<\cdots<x_{d-1}=\hat{1}$ is a fundamental chain of $[\alpha,\hat{1}]$. If $j=d-2$ and $\alpha\neq x_1$, then either $\alpha<x_1\vee\alpha<\cdots<x_{d-3}\vee\alpha<x_{d-1}=\hat{1}$ or $\alpha<x_1\vee\alpha<\cdots<x_{d-4}\vee\alpha<x_{d-1}=\hat{1}$ is a fundamental chain of $[\alpha,\hat{1}]$ since $\mu_L(x_{d-5}\vee\alpha,x_{d-1})=0$ by Lemma (2.2). ∎

We are now in the position to give a proof of Theorem (0.1). Let L be a modular lattice of rank $d-1$ (≥ 3) and suppose that there exists a closed interval $I=[x,y]$ of L with $\mathrm{rank}(I)=3$ such that $\mu_L(x,y)\neq 0$, i.e., I is atomic. We employ induction on $\mathrm{rank}(L)$ to show that the comparability graph $\mathrm{Com}(L)$ of L is d-connected. If $\mathrm{rank}(L)=3$, then $\mu_L(\hat{0},\hat{1})\neq 0$. Thus, L is a geometric lattice and the chain $\hat{0}<\hat{1}$ of length 1 is the only fundamental chain of L. Hence, $\mathrm{Com}(L)$ is 4-connected by Corollary (1.4). If $\mathrm{rank}(L)=4$, then Lemma (2.2) guarantees that there exists no fundamental chain of length 3 of L. Hence, by Corollary (1.4) again, $\mathrm{Com}(L)$ is 5-connected.

Let us now assume $\mathrm{rank}(L)\geq 5$ and suppose that the comparability graph $\mathrm{Com}(L)$ of L is *not* d-connected. Then, Corollary (1.4) guarantees the existence of a fundamental chain of length $d-2$ of L. Hence, by Lemma (2.3), if α is an atom of L, then the closed interval $[\alpha,\hat{1}]$ of L possesses a fundamental chain of length $\geq d-3$. If there exists a fundamental chain of $[\alpha,\hat{1}]$ of length $d-2$, then $[\alpha,\hat{1}]$ is a totally ordered set and no closed interval I of L with $I\subset[\alpha,\hat{1}]$ and with $\mathrm{rank}(I)=3$ is atomic. If there exists a fundamental chain of $[\alpha,\hat{1}]$ of length $d-3$, then our assumption of induction together with Corollary (1.4) says that no closed interval I of L with $I\subset[\alpha,\hat{1}]$ and with $\mathrm{rank}(I)=3$ is atomic. Hence, for an arbitrary atom α of L, no closed interval I of L with $I\subset[\alpha,\hat{1}]$ and with $\mathrm{rank}(I)=3$ is atomic. On the other hand, by Lemma (2.1;b), no closed interval I of L of the form $I=[\hat{0},\delta]$ with $\mathrm{rank}_L(\delta)=3$ is atomic. Thus, if L possesses a fundamental chain of length $d-2$, then no closed interval I of L with $\mathrm{rank}(I)=3$ is atomic. Hence, the existence of a closed interval $I=[x,y]$ of L with $\mathrm{rank}(I)=3$ such that $\mu_L(x,y)\neq 0$ guarantees that the comparability graph $\mathrm{Com}(L)$ of L is d-connected as required. ∎

(2.4) Example. Let L be the semimodular lattice of rank 4 drawn below. Then, the closed interval $I = [x, \hat{1}]$ of L with $\text{rank}(I) = 3$ is a boolean algebra. However, the comparability graph $\text{Com}(L)$ of L is not 5-connected, since the subgraph $\text{Com}(L - \{\hat{0}, \hat{1}, \alpha, \beta\})$ is disconnected. Hence, Theorem (0.1) is, in general, false for a finite semimodular lattice.



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